Class 18, given on Feb 11, 2010, for Math 13, Winter 2010

1. Conservative vector fields

Let $\mathbf{F}$ be a C^1 vector field defined on an open, connected region D. Recall the following properties are equivalent:

- $\mathbf{F}$ is conservative on D; ie, $\mathbf{F} = \nabla f$ for some differentiable potential function $f(x, y)$ defined on D.
- $\mathbf{F}$ is path-independent on D.
- $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C which lies entirely in D.

Furthermore, if $\mathbf{F} = \langle P, Q\rangle$ is conservative on D, then $P_y = Q_x$.

If D is simply-connected (intuitively, no holes), and if $P_y = Q_x$ on all of D, then $\mathbf{F}$ is conservative.

Examples.

- Let $\mathbf{F}(x, y) = \langle 3x^2, 3y^2\rangle$. Use the criterion $P_y = Q_x$ to show that $\mathbf{F}$ is conservative, and then find a potential function for $\mathbf{F}$.

  Since $P = 3x^2, Q = 3y^2, P_y = Q_x = 0$. Furthermore, $\mathbf{F}$ is defined for all of $\mathbb{R}^2$, which is simply connected, so $\mathbf{F}$ is conservative.

  To find a potential function f, we use 'partial integration'. Since $f_x = 3x^2, f(x, y) = x^3 + g(y)$, for some function $g(y)$. Similarly, because $f_y = 3y^2, f(x, y) = y^3 + h(x)$, for some function $h(x)$. Since $x^3 + g(y) = y^3 + h(x)$, we pick $g(y) = y^3, h(x) = x^3$, so $f(x, y) = x^3 + y^3$.
- Let $\mathbf{F}(x, y) = \langle \cos y + ye^{xy}, -x \sin y + xe^{xy}\rangle$. Use the criterion $P_y = Q_x$ to show that $\mathbf{F}$ is conservative, and then find a potential function for $\mathbf{F}$.

  Since $P(x, y) = \cos y + ye^{xy}, P_y = -\sin y + e^{xy} + xye^{xy}$. Similarly, $Q_x = -\sin y + e^{xy} + xye^{xy}$. These are equal; furthermore, $\mathbf{F}$ is C^1 on all of $\mathbb{R}^2$, which is simply connected, so we can conclude that $\mathbf{F}$ is conservative.

  To find a potential function f, we use 'partial integration'. Since $f_x = \cos y + ye^{xy}, f(x, y) = x \cos y + e^{xy} + g(y)$. Similarly, because $f_y = -x \sin y + xe^{xy}, f(x, y) = x \cos y + e^{xy} + h(x)$. Therefore, $f(x, y) = x \cos y + e^{xy}$ is a potential function for $\mathbf{F}$.

2. Green's Theorem

We now discuss a theorem which connects double integrals with line integrals. Let C be a simple closed curve lying in $\mathbb{R}^2$. Then the *positive orientation* of C is defined to be the orientation of C we obtain by moving in single counterclockwise loop along C. An alternate definition is that if we walk in the direction of the positive orientation for C, the interior of C is always on our left-hand side. The *negative orientation* of C is the opposite orientation of the positive orientation for C.

Theorem. (Green's Theorem) Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$. Let D be the region bounded by C. If $\mathbf{F} = \langle P, Q\rangle$ is a C^1 vector field on D, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

An alternate formulation of Green's Theorem is

$$\int_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

- We will not prove Green's Theorem, or give an indication of how the proof goes yet, but we will see how it is a consequence of a more general theorem we will study in a few weeks.
- Green's Theorem should be philosophically interpreted as a higher-dimensional analogue of the FTC. The usual FTC relates the value of a double integral over an interval to the value of its antiderivative at the endpoints, which is also the boundary, of that interval. Green's Theorem relates the value of the double integral of the expression $Q_x - P_y$ over a region D to a line integral of a related function (which may not look like an antiderivative, but certainly involves 'partial integrals' of Q_x, P_y) over the boundary of D.
- Green's Theorem can sometimes act as a bridge between line integrals and double integrals. For example, it may be difficult or tedious to evaluate a certain line integral, but an application of Green's Theorem might convert that line integral into a double integral which is easier to evaluate. Conversely, a double integral which might look difficult to evaluate can sometimes be converted to a line integral which is easier to evaluate, although this is slightly more difficult to do.
- The strategy of when to use Green's Theorem: In general, if you have to evaluate a line integral over a rectangle, or the boundary of some other simple two-dimensional region which consists of several different pieces, using Green's Theorem will usually simplify your calculation. Calculating a line integral over a rectangle involves breaking up that rectangle into its four sides, separately calculating parameterizations for each side, and then calculating four different line integrals. However, an application of Green's Theorem will convert the line integral into a single double integral over a rectangle, which usually is easy to do.

  Line integrals over regions like circles may or may not be simplified using Green's Theorem. It depends on the vector field being integrated. If you find you are having difficulty evaluating a certain line integral because the resulting integrand is excessively complicated, try using Green's Theorem to see if you get a simple double integral.

Examples.

- Let $\mathbf{F} = \langle y\cos x, x^2\rangle$, and let C be the boundary of the square R, given by $0 \le x \le 1, 0 \le y \le 1$ with positive orientation. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

  If you wanted to, you could split C up into its four sides, parameterize each side, and then evaluate the line integral along each side, but that's a lot of work. If you apply Green's Theorem, with $P = y\cos x, P_y = \cos x, Q = x^2, Q_x = 2x$, then we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_R Q_x - P_y\,dA = \int_0^1\int_0^1 2x - \cos x\,dy\,dx = \int_0^1 2x - \cos x\,dx = x^2 - \sin x\Big|_0^1 = 1 - \sin 1.$$

- Let $\mathbf{F} = \langle -y^3 + \log(\sin x), x^3 + \arctan y\rangle$, and let C be the circle $x^2 + y^2 = 1$ with counterclockwise orientation. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

  If you try to directly calculate this line integral, you will have a really difficult time because the resulting integral in the parameter t will be a complete mess. If you try to evaluate this integral using the fundamental theorem for line integrals you will also fail, because $\mathbf{F}$ is not conservative. Therefore, you should try Green's Theorem.

Since $P = -y^3 + \log(\sin x), P_y = -3y^2$. Similarly, $Q_x = 3x^2$, so Green's Theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_D Q_x - P_y \, dA = \iint\limits_D 3x^2 + 3y^2 \, dA,$$

where D is the disc $x^2 + y^2 \leq 1$. This looks like an integral we should evaluate using polar coordinates. If we convert this double integral to polar coordinates we get

$$\iint\limits_D 3x^2 + 3y^2 \, dA = \int_0^{2\pi} \int_0^1 3r^2 r \, dr \, d\theta = \int_0^{2\pi} 3/4 \, d\theta = \frac{3\pi}{2}.$$

- An interesting application of Green's Theorem is to the calculation of areas of two-dimensional regions. Recall that the area of a region D can be expressed as the value of the double integral $\iint\limits_D dA$. If we select P, Q such that $Q_x - P_y = 1$, then $\int_C P \, dx + Q \, dy$ will also equal the area of D, where C is the boundary of D with positive orientation.

  For example, selecting $Q = x, P = 0$ yields the equation $A(D) = \int_C x \, dy$. Selecting $Q = 0, P = -y$, or $Q = x/2, P = -y/2$ gives the equations

$$A(D) = -\int_C y \, dx = \frac{1}{2} \int_C x \, dy - y \, dx.$$

  Let us apply this to the calculation of the area of an ellipse (which we already know how to do using other means). Suppose D is the region $x^2/a^2 + y^2/b^2 \leq 1$; this is the region enclosed by an ellipse with axes of length $2a, 2b$. The boundary C of D is parameterized by $\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle, 0 \leq t \leq 2\pi$. If we use the last of the expressions for the area of D, we get

$$A(D) = \frac{1}{2} \int_C x \, dy - y \, dx = \frac{1}{2} \int_0^{2\pi} (a \cos t)(b \cos t) - (b \sin t)(-a \sin t) \, dt = \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab.$$

  The homework exercises also have a very interesting application of this formula to finding the area of polygons.

- Suppose $\mathbf{F} = \langle P, Q \rangle$ is conservative on D. Then $P_y = Q_x$, so an application of Green's Theorem gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_D Q_x - P_y \, dA = \iint\limits_D 0 \, dA = 0.$$

  This is exactly as we expect by the FTC for line integrals, so in some sense Green's Theorem is a generalization of the FTC for line integrals, at least for regions D enclosed by simple closed curves.